

Expansiveness of algebraic actions on connected groups

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1 Introduction

Let (X, d) be a metric space and ρ be a continuous action of a semigroup Γ on X . Then (X, ρ) is said to be *expansive* if there exists an $\epsilon > 0$ such that for any two distinct points x, y in X ,

$$\sup_{\gamma \in \Gamma} d(\rho(\gamma)(x), \rho(\gamma)(y)) \geq \epsilon.$$

Any such ϵ is called an expansive constant of (X, ρ) . It is easy to check that when X is compact, the notion of expansiveness is independent of the metric d .

If X is a metrizable topological group and ρ is an action of a semigroup Γ on X by continuous endomorphisms of X , then ρ is said to be *expansive* if there exists a neighborhood U of the identity in X such that

$$\bigcap_{\gamma} \rho(\gamma)^{-1}(U) = \{e\}.$$

Any such neighborhood is called an *expansive neighborhood* of the identity in X . An automorphism τ of X is said to be expansive if the cyclic group generated by τ acts expansively on X . It is easy to check that when X is compact and ρ is an endomorphism action of a semigroup Γ , these two notions of expansiveness coincide.

The notion of expansiveness plays an important role in the study of dynamical systems in general and endomorphism actions in particular. Earlier, expansiveness of automorphisms of connected groups was studied by several

authors (cf. [1], [2], [6], [7] and [13]). A complete description of expansive automorphisms on compact connected groups was obtained in [13]. In particular, it was shown that every compact connected group which admits an expansive automorphism is abelian and finite-dimensional. Also in [12] it was proved that if a compact connected group X admits an expansive endomorphism action then it is abelian.

In recent years dynamics of endomorphism actions of discrete semigroups on compact abelian groups has been extensively studied using techniques from commutative algebra (cf. [18]). Among other things, a complete characterization of expansiveness has been obtained when X is zero-dimensional (cf. [10], Theorem 5.2) or $\Gamma = \mathbb{Z}^d$ for some $d \geq 1$ (cf. [17], Theorem 3.9). For further study of expansiveness and its relation with other dynamical properties the reader is referred to [4], [9] and [14].

In this paper we consider expansive endomorphism actions of arbitrary semigroups on connected metrizable topological groups. For any such action (G, ρ) we give necessary and sufficient conditions for expansiveness of ρ , provided G is either a Lie group or a compact finite-dimensional group.

This paper is organized as follows. In section 2 we prove some elementary results about expansiveness of endomorphism actions on finite-dimensional vector spaces, considered as abelian topological groups under addition. In Section 3 we consider endomorphism actions on connected Lie groups. If G is a connected Lie group then by $L(G)$ we denote the Lie algebra of G . If ρ is an endomorphism action of a semigroup Γ on G then by ρ_e we denote the induced Γ -action on $L(G)$ defined by

$$\rho_e(\gamma) = d\rho(\gamma)|_e \quad \forall \gamma \in \Gamma.$$

We prove the following.

Theorem A. *Let G be a connected Lie group, Γ be a semigroup and ρ be an endomorphism action of Γ on G . Then (G, ρ) is expansive if and only if for all non-zero v in $L(G)$, the ρ_e -orbit of v is unbounded.*

In the special case when Γ is either abelian or a virtually nilpotent group, applying the above theorem we give a necessary and sufficient condition for expansiveness of (G, ρ) in terms of the generalized weights of ρ_e . As a consequence we prove that if Γ is a virtually nilpotent group and (G, ρ) is expansive then for some γ_0 in Γ , $\rho(\gamma_0)$ is an expansive automorphism of G .

In section 4 we consider endomorphism actions on compact connected finite-dimensional abelian groups. For any such group G , by \widehat{G} we denote the Pontryagin dual of G and by $L(G)$ we denote the vector space of all homomorphisms from \widehat{G} to \mathbb{R} under pointwise addition and scalar multiplication. From the duality theory of compact abelian groups it follows that $L(G)$ is finite-dimensional. If ρ is an endomorphism action of a semigroup Γ on G then by ρ_e we denote the induced Γ -action on $L(G)$ defined by

$$\rho_e(\gamma)(p)(\chi) = p(\chi \circ \rho(\gamma)) \quad \forall p \in L(G).$$

Also, for such an action ρ , by $\widehat{\rho}$ we denote the induced Γ -action on \widehat{G} . We note that \widehat{G} can be realised as a module over $\mathbb{Z}(\Gamma)$, the group-ring of Γ , via the action $\widehat{\rho}$.

We prove the following.

Theorem B. *Let Γ be a semigroup and ρ be an endomorphism action of Γ on a compact connected finite-dimensional abelian group G . Then (G, ρ) is expansive if and only if the following two conditions are satisfied.*

- a) \widehat{G} is finitely generated as a $\mathbb{Z}[\Gamma]$ -module.
- b) For every non-zero v in $L(G)$, the ρ_e -orbit of v is unbounded.

Remark 1.1 Since non-abelian compact connected groups do not admit expansive endomorphism actions (cf. [12]), Theorem B characterizes expansive endomorphism actions on arbitrary compact connected finite-dimensional groups.

2 Preliminaries

Throughout this paper Γ will denote a discrete semigroup i.e. a set with an associative multiplication law. If Γ_1, Γ_2 are semigroups then a map f from Γ_1 to Γ_2 is said to be a semigroup homomorphism if for all γ_1, γ_2 in Γ_1 , $f(\gamma_1 \cdot \gamma_2) = f(\gamma_1) \cdot f(\gamma_2)$. If G is a topological group then by $\text{End}(G)$ we denote semigroup of all continuous endomorphisms of G , with composition as the product law.

Suppose G is a topological group and $\rho : \Gamma \rightarrow \text{End}(G)$ is an endomorphism action of a discrete semigroup Γ on G . Then it is easy to see that $\Gamma_1 =$

$\rho(\Gamma) \cup \{\text{Id}\}$ is a subsemigroup of $\text{End}(G)$ and ρ is expansive if and only if the natural action of Γ_1 on G is expansive. Henceforth we will restrict the discussion of endomorphism actions to the case when Γ contains an identity element e and $\rho(e) \in \text{End}(G)$ is the identity automorphism.

If \mathbf{C} denotes the semigroup of complex numbers under multiplication then any semigroup homomorphism $\lambda : \Gamma \rightarrow \mathbf{C}$ is said to be a *character* of Γ . If $\rho : \Gamma \rightarrow \text{End}(V)$ is an endomorphism action of Γ on a finite-dimensional vector space V over \mathbb{C} , then ρ is said to be *reducible* if there exists characters $\lambda_1, \dots, \lambda_k$ and nontrivial ρ -invariant subspaces $V_1, \dots, V_k \subset V$ satisfying the following conditions.

1. $V = V_1 \oplus \dots \oplus V_k$.
2. For $i = 1, \dots, k$, there exists a basis B of V_i such that $\rho(\gamma)|_{V_i} - \lambda_i(\gamma)I$ is strictly upper triangular with respect to B , for all γ in Γ .

The characters $\lambda_1, \dots, \lambda_k$ are said to be the *generalized weights* of ρ and for all i , V_i is said to be the *generalized weight space* corresponding to λ_i .

Example 2.1 If $\rho(\Gamma)$ is an abelian subsemigroup $\text{End}(V)$ then ρ is reducible (see [8], pp 134). This is also true when $\rho(\Gamma)$ is contained in a connected nilpotent subgroup H of $GL(V)$. This can be seen by looking at the Lie algebra homomorphism $i : L(H) \rightarrow \text{End}(V)$ and using certain fundamental facts about representations of nilpotent Lie algebras (see [11], Proposition 2.4 and Theorem 1.35).

The following proposition gives a necessary and sufficient condition for expansiveness of endomorphism actions on a finite-dimensional vector space, considered as an abelian group under addition.

Proposition 2.2 *Let Γ be a semigroup and $\rho : \Gamma \rightarrow \text{End}(V)$ be an endomorphism action of Γ on a finite-dimensional vector space V over \mathbb{R} or \mathbb{C} . Then we have the following :*

- a) ρ is expansive if and only if for all non-zero v in V , the Γ -orbit of v is unbounded.
- b) If V is a real vector space then ρ is expansive if and only if the induced endomorphism action of Γ on $V \otimes \mathbb{C}$ is expansive.

Proof For any v in V let $O_v \subset V$ denote the ρ -orbit of v . Suppose that there is a non-zero vector v in V with bounded ρ -orbit. Let U be any neighborhood of 0 in V . Then for $\epsilon > 0$ sufficiently small, $O_{\epsilon v} = \epsilon O_v$ is contained in U . This implies that ρ is not expansive. On the other hand if O_v is unbounded for all non-zero $v \in V$, then it is easy to see that any bounded neighborhood of 0 is an expansive neighborhood. This proves a). Part b) is an immediate consequence of a). \square

Proposition 2.3 *Let Γ be a semigroup, V be a finite-dimensional vector space over \mathbb{C} and $\rho : \Gamma \rightarrow \text{End}(V)$ be a reducible endomorphism action with generalized weights $\lambda_1, \dots, \lambda_k$. Then ρ is expansive if and only if for each i , the image of λ_i is an unbounded subset of \mathbb{C} .*

Proof For $i = 1, \dots, k$ let V_i be the generalized weight space corresponding to λ_i . Since each V_i is ρ -invariant, from the previous proposition it follows that ρ is expansive if and only if $\rho|_{V_i}$ is expansive for all i . Therefore without loss of generality we may assume that $k = 1$. In that case there exists a character λ of Γ such that after suitable identifications we have :

- a) $V = \mathbb{C}^n$ for some $n > 0$ and
- b) For all $\gamma \in \Gamma$, $\rho(\gamma)$ can be written as,

$$\rho(\gamma) = \begin{pmatrix} \lambda(\gamma) & & & \\ & \ddots & & * \\ & & \lambda(\gamma) & \\ & 0 & & \ddots \\ & & & & \lambda(\gamma) \end{pmatrix}.$$

Suppose there exists a non-zero vector $v = (v_1, \dots, v_n)^t$ in \mathbb{C}^n such that the ρ -orbit of v is bounded. Let $j \leq n$ be the largest positive integer such that $v_j \neq 0$. Then it is easy to see that

$$\{\rho(\gamma)(v)_j \mid \gamma \in \Gamma\} = \{\lambda(\gamma)v_j \mid \gamma \in \Gamma\}.$$

Since the ρ -orbit of v is bounded, this implies that the image of λ is bounded subset of \mathbb{C} . Conversely, it is easy to see that if $v \in \mathbb{C}^n$ denotes the vector $(1, 0, \dots, 0)^t$ then

$$\{\rho(\gamma)(v) \mid \gamma \in \Gamma\} = \{(\lambda(\gamma), 0, \dots, 0)^t \mid \gamma \in \Gamma\}.$$

Therefore if the image of λ is bounded then there exists a non-zero vector in V with bounded ρ -orbit. \square

In the special case when Γ is isomorphic to \mathbb{Z}^+ or \mathbb{Z} we obtain the following consequence of the above proposition. This was first proved in [7].

Proposition 2.4 *Let V be a finite-dimensional vector space over \mathbb{R} or \mathbb{C} and let T be an element of $\text{End}(V)$. Then the cyclic semigroup generated by T acts expansively on V if and only if the spectrum of T does not intersect the closed unit disk. If T is invertible then the cyclic subgroup generated by T acts expansively on V if and only if the spectrum of T does not intersect \mathbb{S}^1 .*

Proof Replacing V by $V \otimes \mathbb{C}$ if necessary, we may assume that V is a complex vector space. Let $\Gamma = \{T^n \mid n \geq 0\}$ denote the cyclic semigroup generated by T and let ρ denote the natural action of Γ on V . Since Γ is abelian, ρ is reducible. Furthermore if $\lambda_1, \dots, \lambda_k$ denote the generalized weights of ρ then $\lambda_1(T), \dots, \lambda_k(T)$ are the eigenvalues of T . Applying the previous proposition we conclude that ρ is expansive if and only if the spectrum of T does not intersect the closed unit disk. The second assertion follows from a similar argument. \square

3 Expansive actions on Lie groups

In this section we prove Theorem A and its consequences. Throughout the section G will denote a connected Lie group and $L(G)$ will denote the Lie algebra of G . The standard exponential map from $L(G)$ to G will be denoted by \exp .

We begin with the following lemma.

Lemma 3.1 *Let V be a finite-dimensional vector space over \mathbb{R} and let ρ be an expansive endomorphism action of a semigroup Γ on V . Then there exists a finitely generated subsemigroup $\Gamma_0 \subset \Gamma$ which acts expansively on V .*

Proof Suppose this is not the case. We fix a norm $\|\cdot\|$ on V . Let $S \subset V$ denote the set defined by $S = \{v \in V \mid \|v\| = 1\}$. For every finitely generated

subsemigroup $\Lambda \subset \Gamma$ we define a closed subset $S(\Lambda) \subset S$ by

$$S(\Lambda) = \{v \in S \mid \|\rho(\gamma)(v)\| \leq 1 \ \forall \gamma \in \Lambda\}.$$

We claim that for any finitely generated subsemigroup $\Lambda \subset \Gamma$, $S(\Lambda)$ is non-empty. To see this we define a Λ -invariant subspace $V(\Lambda) \subset V$ by

$$V(\Lambda) = \{v \in V \mid \Lambda\text{-orbit of } v \text{ is bounded}\}.$$

By Proposition 2.2, $V(\Lambda)$ is a non-trivial subspace of V . Let $\Lambda_b \subset \text{End}(V(\Lambda))$ be the image of Λ under the map $\gamma \mapsto \rho(\gamma)|_{V(\Lambda)}$. Then $\overline{\Lambda_b}$ is a compact subsemigroup of $\text{End}(V(\Lambda))$. We choose a non-zero $w \in V(\Lambda)$ and define a continuous function $h : \overline{\Lambda_b} \rightarrow \mathbb{R}$ by $h(\alpha) = \|\alpha(w)\|$ for all α in $\overline{\Lambda_b}$. Since $\overline{\Lambda_b}$ is compact, there exists a $\alpha_0 \in \overline{\Lambda_b}$ such that $h(\alpha_0) \geq h(\alpha)$ for all α in $\overline{\Lambda_b}$. Putting $v = \alpha_0(w)$ we see that for any α in $\overline{\Lambda_b}$,

$$\|\alpha(v)\| = h(\alpha\alpha_0) \leq h(\alpha_0) = \|v\|.$$

Since $\overline{\Lambda_b}$ contains Id , $\|\alpha_0(w)\| \geq \|w\| > 0$. Now it is easy to see that the unit vector $v/\|v\| \in S(\Lambda)$, which proves the claim.

If $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ are finitely generated subsemigroups of Γ and Λ is a finitely generated subsemigroup of Γ containing $\Lambda_1, \Lambda_2, \dots, \Lambda_k$, then from the above claim it follows that

$$\emptyset \neq S(\Lambda) \subset S(\Lambda_1) \cap \dots \cap S(\Lambda_k).$$

This shows that the collection $\{S(\Lambda)\}$ has the finite-intersection property. Now from the compactness of S it follows that

$$\bigcap_{\Lambda} S(\Lambda) \neq \emptyset.$$

Clearly for any vector w which lies in the intersection, the Γ -orbit of w is bounded, which contradicts our hypothesis. \square

Proof of Theorem A. Suppose (G, ρ) is expansive. Let U be an expansive neighborhood of the identity in G . We choose a neighborhood V of 0 in $L(G)$ such that $\exp|_V$ is a homeomorphism and $\exp(V)$ is contained in U . Let v be any non-zero element of $L(G)$ such that the ρ_e -orbit of v is

contained in V . Since \exp is a Γ -equivariant map from $(L(G), \rho_e)$ to (G, ρ) , it follows that the ρ -orbit of $\exp(v)$ is contained in U . Since U is an expansive neighborhood of the identity and $\exp|_V$ is a homeomorphism, this implies that $v = 0$. Hence ρ_e is expansive. Now applying Proposition 2.2 we see that for every non-zero v in $L(G)$, the ρ_e -orbit of v is unbounded.

Now we will prove the converse. Applying Lemma 3.1 and Proposition 2.2 we see that it is enough to consider the case when Γ is finitely generated. Let A be a finite set which generates Γ . We choose a neighborhood V of 0 in $L(G)$ such that $\exp|_V$ is a homeomorphism and \overline{V} is compact. We choose another neighborhood U of 0 such that

$$U \subset V \text{ and } \rho_e(\gamma)(U) \subset V \text{ } \forall \gamma \in A.$$

We claim that $\exp(U)$ is an expansive neighborhood of (G, ρ) . Suppose this is not the case. Let $g \neq e$ be any element of G such that the ρ -orbit of g is contained in $\exp(U)$. We choose v in U such that $\exp(v) = g$. Let O_v be the ρ_e -orbit of v . Since v is non-zero, O_v is unbounded. Since A generates Γ and $v \in U$, it follows that there exists w in O_v and $\gamma_0 \in A$ such that

$$w \in U \text{ and } \rho_e(\gamma_0)(w) \in V - U.$$

Since \exp maps the ρ_e -orbit of v onto the ρ -orbit of $\exp(v)$, this implies that the ρ -orbit of $\exp(v)$ intersects $\exp(V - U)$. Since $\exp(V - U)$ and $\exp(U)$ are disjoint, this gives a contradiction. \square

We note the following consequence of Theorem A and Lemma 3.1.

Corollary 3.2 *Let Γ be a discrete semigroup, G be a connected Lie group and ρ be an expansive endomorphism action of Γ on G . Then there exists a finitely generated subsemigroup $\Gamma_0 \subset \Gamma$ such that the action of ρ , restricted to Γ_0 , is expansive.*

The following corollary provides a rich source of expansive endomorphism actions of non-abelian semigroups on tori.

Corollary 3.3 *Let Γ be an infinite subsemigroup of $M(n, \mathbb{Z})$ which acts irreducibly on \mathbb{R}^n . Then the induced Γ -action on $\mathbb{T}^n \cong \mathbb{R}^n / \mathbb{Z}^n$ is expansive.*

Proof Let ρ denote the natural action of Γ on \mathbb{R}^n and W denote the subspace of \mathbb{R}^n consisting of all points with bounded ρ -orbit. Since Γ is an infinite subsemigroup of $M(n, \mathbb{Z})$, it is a noncompact subset of $M(n, \mathbb{R})$. Hence W is a proper subspace of \mathbb{R}^n . From irreducibility of ρ we conclude that $W = \{0\}$. Now applying Theorem A we see that the action of Γ on \mathbb{T}^n is expansive. \square

Corollary 3.4 *Let G be a connected Lie group and ρ be an expansive automorphism action of a virtually nilpotent group Γ on G . Then $\rho(\Gamma)$ contains an expansive automorphism of G .*

Proof Since Γ acts expansively on G , so does every finite-index subgroup of Γ . Therefore without loss of generality we may assume that Γ is nilpotent. First we will consider the case when G is isomorphic to a finite-dimensional vector V space over \mathbb{C} . Let $H \subset GL(V)$ be the Zariski-closure of $\rho(\Gamma)$ and H_0 be the connected component of H which contains the identity. Since H has finitely many components, it follows that

$$\Gamma_0 = \{\gamma \in \Gamma \mid \rho(\gamma) \in H_0\}$$

is a finite-index subgroup of Γ . Let ρ_0 denote the Γ_0 -action on V , induced by ρ . Since $\rho(\Gamma_0)$ is contained in a connected nilpotent subgroup of $GL(V)$, ρ_0 is reducible. Since ρ is expansive, so is ρ_0 . Let $\lambda_1, \dots, \lambda_k$ be the generalized weights of ρ_0 . For each γ in Γ_0 we define $A_\gamma \subset \{1, \dots, k\}$ by

$$A_\gamma = \{j \mid |\lambda_j(\gamma)| \neq 1\}.$$

For any $\gamma \in \Gamma_0$ let $n(\gamma)$ denote the cardinality of A_γ . We choose $\gamma_0 \in \Gamma_0$ such that the $n(\gamma_0) \geq n(\gamma)$ for all γ in Γ_0 . We claim that $n(\gamma_0) = k$ i.e. $|\lambda_i(\gamma_0)| \neq 1$ for all $j = 1, \dots, k$. Suppose this is not the case. We choose i such that $|\lambda_i(\gamma_0)| = 1$. By Proposition 2.3 there exists γ_1 such that $|\lambda_i(\gamma_1)| \neq 1$. It is easy to see that for sufficiently large $m > 0$,

$$|\lambda_j(\gamma_0^m \gamma_1)| \neq 1 \quad \forall j \in A_{\gamma_0} \cup \{i\}.$$

Since $n(\gamma_0) \geq n(\gamma)$ for all γ in Γ_0 , this gives a contradiction. Since the numbers $\lambda_1(\gamma_0), \dots, \lambda_k(\gamma_0)$ are the eigenvalues of $\rho(\gamma_0)$, from the above claim and Proposition 2.4 we see that $\rho(\gamma_0)$ is an expansive automorphism of V .

Now we will consider the general case. Let σ be the Γ -action on the complexified Lie algebra $L(G) \otimes \mathbb{C}$, induced by ρ_e . Applying Theorem A and Proposition 2.2 we see that σ is expansive. By the previous argument there exists a γ_0 in Γ such that $\sigma(\gamma_0)$ is an expansive automorphism of $L(G) \otimes \mathbb{C}$. Applying Theorem A and Proposition 2.2 to the cyclic group generated by γ_0 , we conclude that $\rho(\gamma_0)$ is an expansive automorphism of G . \square

It is known that if a connected locally compact topological group admits an expansive automorphism then it is nilpotent (see [1], [15]). The following corollary generalises this result for actions of virtually nilpotent groups on connected Lie groups.

Corollary 3.5 *Let G be a connected Lie group which admits an expansive automorphism action of a virtually nilpotent group. Then G is nilpotent.*

Proof Let ρ be an automorphism action of a virtually nilpotent group Γ on G . Then by the previous corollary there exists a $\gamma \in \Gamma$ such that $\rho(\gamma)$ is an expansive automorphism of G . Applying Theorem A and Proposition 2.4 we see that the spectrum of $d\gamma \in \text{Aut}(L(G))$ does not intersect the unit circle. It is known that if a finite-dimensional Lie algebra over \mathbb{R} admits a hyperbolic automorphism then it is nilpotent. Hence G is nilpotent. \square

We conclude this section with an example showing that Corollary 3.4 does not hold for actions of arbitrary discrete groups.

Example 3.6 Fix $n \geq 3$ and define a subgroup Γ of $GL(n, \mathbb{R})$ by

$$\Gamma = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n-1, \mathbb{Z}), b \in \mathbb{Z}^{n-1} \right\}$$

It is easy to see that for any $x = (x_1, \dots, x_n)$ in \mathbb{R}^n , the Γ -orbit of x is given by the set

$$\{Ay + x_nb \mid A \in GL(n-1, \mathbb{Z}), b \in \mathbb{Z}^{n-1}\}; \quad y = (x_1, \dots, x_{n-1}).$$

Hence for every non-zero x in \mathbb{R}^n , the Γ -orbit of x is unbounded. Since the spectrum of γ contains 1 for every γ in Γ , applying Proposition 2.4 we see that Γ does not contain any expansive automorphism of \mathbb{R}^n .

4 Expansive actions on solenoids

In this section we consider endomorphism actions on solenoids (compact connected finite-dimensional abelian groups). We freely use various results from duality theory of locally compact abelian groups; the reader is referred to [16] for details.

For any locally compact abelian group G , we denote by \widehat{G} the dual group of G . Recall that for a compact connected abelian group G , we denote by $L(G)$ the vector space consisting of all homomorphisms from \widehat{G} to \mathbb{R} , under pointwise addition and scalar multiplication. It is known that if G is a solenoid, then \widehat{G} is a torsion free discrete abelian group of finite rank. Hence for any solenoid G , $L(G)$ is finite-dimensional.

If ρ is an endomorphism action of a semigroup Γ on a compact connected abelian group G then by $\widehat{\rho}$ we denote the induced Γ -action on \widehat{G} and by ρ_e we denote the induced Γ -action on $L(G)$ defined by

$$\rho_e(\gamma)(p)(\chi) = p(\chi \circ \rho(\gamma)).$$

Example 4.1 Let Γ be a semigroup, $\widehat{\rho} : \Gamma \rightarrow M(n, \mathbb{Q})$ be a semigroup homomorphism for some $n \geq 1$ and H be a Γ -invariant subgroup of \mathbb{Q}^n . Then the action of Γ on H induces an endomorphism action ρ of Γ on $G = \widehat{H}$. If H contains \mathbb{Z}^n then $L(G)$ can be identified with \mathbb{R}^n and ρ_e can be identified with the adjoint of the Γ -action on \mathbb{R}^n induced by $\widehat{\rho}$. From duality theory of compact abelian groups it follows that any endomorphism action of a discrete semigroup on a solenoid can be identified with an action of this form.

This section is organized as follows. Throughout this section G will denote a solenoid and ρ will denote an endomorphism action of a discrete semigroup Γ on G . In 4.1 we prove that the conditions a) and b), as stated in Theorem B are necessary for expansiveness of ρ . In 4.2 we show that expansiveness of ρ and existence of non-trivial bounded ρ_e -orbit in $L(G)$ can be characterized in terms of suitably chosen metrics on G and $L(G)$ respectively. In 4.3 we prove Theorem B when Γ is finitely generated. We complete the proof of Theorem B in 4.4.

4.1 Necessary conditions for expansiveness

For any compact connected abelian group G , we define a map E from $L(G)$ to G by the condition

$$(\phi \circ E)(p) = e^{2\pi i p(\phi)} \quad \forall p \in L(G), \phi \in \widehat{G}.$$

Since for a fixed p in $L(G)$, the map $\phi \rightarrow e^{2\pi i p(\phi)}$ is a continuous homomorphism from \widehat{G} to \mathbb{S}^1 , by the duality theorem the map E is well defined and unique. From the uniqueness it follows that E is a homomorphism from $L(G)$ (considered as an abelian group under addition) to G . It is easy to check that the kernel of E can be identified with the set of all homomorphisms from \widehat{G} to \mathbb{Z} , which is a discrete subgroup of $L(G)$.

Remark 4.2 Using the duality theorem we can realize $L(G)$ with the set of all one-parameter subgroups of G and E with the map $\alpha \mapsto \alpha(1)$. In particular when G is a torus, $L(G)$ can be identified with \mathbb{R}^n , the Lie algebra of G , and E can be identified with the standard exponential map. However in general the map E is not surjective. In fact from a result of Dixmier (see [5]) it follows that if G is metrizable then E is surjective if and only if \widehat{G} is a free abelian group.

Definition Let Γ be a semigroup and ρ be an endomorphism action of Γ on a compact abelian group G . Then a set $A \subset \widehat{G}$ is said to be a ρ -basis if A generates \widehat{G} as an abelian group and $A = \bigcup_{\gamma} \widehat{\rho}(\gamma)(F)$ for some finite set $F \subset \widehat{G}$.

Clearly \widehat{G} admits a ρ -basis if and only if it is finitely generated as a $\mathbb{Z}(\Gamma)$ -module.

The next two propositions show that for expansive endomorphism actions the conditions a) and b), as stated in Theorem B are satisfied. The results are known, we include the proofs for the sake of completeness.

Proposition 4.3 *Let Γ be a semigroup and ρ be an expansive endomorphism action of Γ on a compact abelian group G . Then \widehat{G} has a ρ -basis.*

Proof Let U be an expansive neighborhood of e in G . We choose a finite set $F \subset \widehat{G}$ and $\epsilon > 0$ such that

$$\bigcap_{\chi \in F} \{g \in G \mid |\chi(g) - 1| < \epsilon\} \subset U.$$

We define $A \subset \widehat{G}$ by $A = \bigcup_{\gamma} \widehat{\rho}(\gamma)(F)$. Let $\widehat{H} \subset \widehat{G}$ be the subgroup generated by A and let $G' \subset G$ be the subgroup consisting of all g in G such that $\chi(g) = e$ for all χ in \widehat{H} . Then for every $g \in G'$, the ρ -orbit of g is contained in U . Since U is an expansive neighborhood of e , we conclude that $G' = \{e\}$. Now from duality theory of compact abelian groups it follows that $\widehat{H} = \widehat{G}$ (see [16], Theorem 53). Hence A is a ρ -basis of \widehat{G} . \square

Proposition 4.4 *(Also see [3], proof of Corollary 1) Let Γ be a semigroup and ρ be an expansive endomorphism action of Γ on a solenoid G . Then for every non-zero point $p \in L(G)$, the orbit of p under ρ_e is unbounded.*

Proof Let U be an expansive neighborhood of e in G . Since E is a Γ -equivariant map from $(L(G), \rho_e)$ to (G, ρ) , it follows that

$$\bigcap_{\gamma \in \Gamma} \rho_e(\gamma)^{-1}(E^{-1}(U)) \subset E^{-1}\left(\bigcap_{\gamma \in \Gamma} \rho(\gamma)^{-1}(U)\right) = \text{Ker}(E).$$

Since the kernel of E is discrete, there exists an open set $V \subset E^{-1}(U)$ such that $V \cap \text{ker}(E) = \{0\}$. From the above identity it is easy to see that V is an expansive neighborhood of 0 for the action ρ_e . Now the given assertion follows from Proposition 2.2. \square

4.2 Metrics induced by endomorphism actions

Let δ be the function from \mathbb{S}^1 to \mathbb{R} defined by $\delta(z) = \inf\{|t| \mid t \in \mathbb{R}, e^{2\pi it} = z\}$. For any solenoid G and $A \subset \widehat{G}$ we define functions $d_A : G \times G \rightarrow [0, 1]$ and $d_A : L(G) \times L(G) \rightarrow [0, \infty]$ by

$$\begin{aligned} d_A(g, h) &= \sup_{\chi \in A} \delta(\chi(g^{-1}h)), \\ d_A^*(p, q) &= \sup_{\chi \in A} |p(\chi) - q(\chi)|. \end{aligned}$$

It is easy to see that if A generates \widehat{G} as an abelian group then d_A is a metric on G and d_A^* is a metric on the subspace $L(G)_A \subset L(G)$ consisting of all p in $L(G)$ with $d_A^*(0, p) < \infty$. For $A \subset \widehat{G}$ and $r > 0$, we define open sets $B_A(r) \subset G$ and $B_A^*(r) \subset L(G)$ by

$$\begin{aligned} B_A(r) &= \{g \in G \mid d_A(e, g) < r\}, \\ B_A^*(r) &= \{p \in L(G) \mid d_A^*(0, p) < r\}. \end{aligned}$$

Suppose ρ is an endomorphism action of a semigroup Γ on a solenoid G and A is a ρ -basis of \widehat{G} . Our next proposition shows that expansiveness of ρ and existence of non-trivial bounded ρ -orbits in $L(G)$ can be characterized in terms of the metrics d_A and d_A^* respectively.

Proposition 4.5 *Let Γ be a semigroup and ρ be an endomorphism action of Γ on a solenoid G . Let A be a ρ -basis of \widehat{G} . Then we have the following.*

1. ρ is expansive if and only if $B_A(\epsilon) = \{e\}$ for some $\epsilon > 0$.
2. ρ_e has a nontrivial bounded orbit if and only if $B_A^*(C) \neq \{0\}$ for some $C > 0$.

Proof 1) Suppose $B_A(\epsilon) = \{e\}$ for some $\epsilon > 0$. We choose a finite set $F \subset A$ such that $A = \bigcup_{\gamma} \widehat{\rho}(\gamma)(F)$. Note that for any g in G , the ρ -orbit of g is contained in $B_F(\epsilon)$ only if $g \in B_A(\epsilon)$. Hence $B_F(\epsilon) \subset G$ is an expansive neighborhood of e for the action ρ .

On the other hand suppose $U \subset G$ is an expansive neighborhood of e . We choose a finite set $F \subset A$ such that $A = \bigcup_{\gamma} \widehat{\rho}(\gamma)(F)$. Since A generates \widehat{G} as an abelian group, it follows that $B_F(\epsilon) \subset U$ for some $\epsilon > 0$. Now it is easy to see that

$$B_A(\epsilon) = \bigcap_{\gamma} \rho(\gamma)^{-1}(B_F(\epsilon)) \subset \bigcap_{\gamma} \rho(\gamma)^{-1}(U) = \{e\}.$$

2) Suppose p is a non-zero element of $L(G)$ such that the ρ_e -orbit of p is bounded. We fix an element ϕ in \widehat{G} and observe that

$$\{q(\phi) \mid q \in \rho_e(\Gamma)(p)\} = \{p(\psi) \mid \psi \in \widehat{\rho}(\Gamma)(\phi)\}.$$

Since $q \mapsto q(\phi)$ is a continuous map from $L(G)$ to \mathbb{R} and the ρ_e -orbit of p is bounded, the right hand side is a bounded subset of \mathbb{R} . Hence p , restricted to the $\widehat{\rho}$ -orbit of ϕ is bounded. Since ϕ is arbitrary and A is a union of finitely many $\widehat{\rho}$ -orbits, we conclude that $p|_A$ is bounded i.e. $p \in B_A^*(C)$ for some $C > 0$.

Conversely, suppose p is an element of $B_A^*(C)$ for some $C > 0$. We fix an element ϕ in \widehat{G} . Since A generates \widehat{G} as an abelian group, there exists a positive integer l and a_1, \dots, a_l in A such that $\phi = a_1 + \dots + a_l$. Since A is invariant under $\widehat{\rho}$, every element in the $\widehat{\rho}$ -orbit of ϕ can be written as sum of l elements of A . Hence $|p(\psi)| < lC$ for all ψ in the $\widehat{\rho}$ -orbit of ϕ i.e. $|q(\phi)| < lC$ for all q in the ρ_e -orbit of p . Since ϕ is arbitrary, this implies that the ρ_e -orbit of p is bounded. \square

4.3 Actions of finitely generated semigroups

As remarked earlier, for an arbitrary solenoid G , the map $E : L(G) \rightarrow G$ need not be surjective. However we will prove the following result which will be a crucial step in the proof of Theorem B.

Theorem 4.6 *Let Γ be a finitely generated semigroup and ρ be an endomorphism action of Γ on a solenoid G . Let A be a ρ -basis of \widehat{G} . Then for any $C > 0$ there exists an $\epsilon > 0$ such that $B_A(\epsilon) \subset E(B_A^*(C))$.*

Note that this theorem, together with Propositions 4.3, 4.4 and 4.5 implies Theorem B when Γ is finitely generated.

Remark 4.7 Earlier we have observed that E is a continuous homomorphism and the kernel of E is discrete. Hence the above theorem implies that the map E , restricted to $L(G)_A$, is a local homeomorphism if $L(G)_A$ and G are equipped with the topologies induced by d_A^* and d_A respectively.

Before beginning the proof of Theorem 4.6 we introduce a few notations.

Let H be a discrete abelian group and A is a finite subset of H . Then by $\mathbf{Q}(A)$ we denote the set of all $h \in H$ which satisfies a linear equation of the form

$$n_0 h = \sum_{j=1}^r n_j a_j,$$

where a_1, \dots, a_r are elements of A and n_0, n_1, \dots, n_r are integers with $n_0 \neq 0$. Also for all $k > 0$, by $\mathbf{Q}_k(A)$ we denote the set of all $h \in H$ which satisfies a linear equation as above with $|n_0| + |n_1| + \dots + |n_r| \leq k$. A set $A \subset H$ is said to be a k -regular if there exists an increasing sequence of finite sets $A_1 \subset A_2 \subset \dots \subset A$ satisfying

$$\bigcup_{i=1}^{\infty} A_i = A, \quad A_n \subset \mathbf{Q}_k(A_{n-1}) \quad \forall n \geq 2.$$

Now we show that if ρ is an endomorphism action of a finitely generated semigroup on a solenoid G , then any ρ -basis is k -regular for some k .

Proposition 4.8 *Let Γ be a finitely generated semigroup and ρ be an endomorphism action of Γ on a solenoid G . Then any ρ -basis of \widehat{G} is k -regular for some $k > 0$.*

Proof Let A be a ρ -basis of \widehat{G} . Since A generates \widehat{G} as an abelian group and \widehat{G} has finite rank, there exists a finite set $F_1 \subset A$ such that $\mathbf{Q}(F_1) = \widehat{G}$. Also, $A = \bigcup_{\gamma} \widehat{\rho}(\gamma)(F_2)$ for some finite set $F_2 \subset A$. We define $F = F_1 \cup F_2$. Let $S \subset \Gamma$ be finite set which generates Γ as a semigroup. Since $\mathbf{Q}(F) = \widehat{G}$, for any γ in S and a_0 in F , $\widehat{\rho}(\gamma)(a_0)$ satisfies a linear equation of the form

$$n_0 \cdot \widehat{\rho}(\gamma)(a_0) = \sum_{j=1}^r n_j a_j,$$

where a_1, \dots, a_r are elements of F and n_0, n_1, \dots, n_r are integers with $n_0 \neq 0$. Since S is finite, it follows that there exists a positive integer k such that $\widehat{\rho}(\gamma)(F) \subset \mathbf{Q}_k(F)$ for all γ in S . For $n \geq 1$ we define $S_n \subset \Gamma$ and $A_n \subset A$ by

$$S_n = \{\gamma_1 \cdots \gamma_n \mid \gamma_1, \dots, \gamma_n \in S\}, \quad A_n = \bigcup_{\gamma \in S_n} \widehat{\rho}(\gamma)(F).$$

Let γ_n be any element of S_n . We choose $\gamma_{n-1} \in S_{n-1}$ and $\gamma_0 \in S$ such that $\gamma_n = \gamma_{n-1} \cdot \gamma_0$. Since $\widehat{\rho}(\gamma_0)(F) \subset \mathbf{Q}_k(F)$, we see that

$$\widehat{\rho}(\gamma_n)(F) \subset \widehat{\rho}(\gamma_{n-1})(\mathbf{Q}_k(F)) \subset \mathbf{Q}_k(\widehat{\rho}(\gamma_{n-1})(F)).$$

This shows that $A_n \subset \mathbf{Q}_k(A_{n-1})$ for all $n \geq 2$. Since S generates Γ , it is easy to see that $\bigcup A_i = \widehat{\rho}(\Gamma)(F) = A$. \square

Now we turn to the proof of Theorem 4.6. We will use the following lemma.

Lemma 4.9 *Let G be a solenoid and F be a finite subset of \widehat{G} . Then for any $C > 0$ there exists an $\epsilon > 0$ such that for any $g \in B_F(\epsilon)$ there exists p in $B_F^*(C)$ satisfying $\phi(g) = \phi \circ E(p)$ for all ϕ in F .*

Proof Let $\widehat{H} \subset \widehat{G}$ be the subgroup generated by F and let H be the dual of \widehat{H} . Since \mathbb{R} is divisible, any homomorphism from \widehat{H} to \mathbb{R} can be extended to a homomorphism from \widehat{G} to \mathbb{R} . Therefore it is enough to consider the case when $G = H$. In that case there exists $n \geq 1$ and an isomorphism $\theta : G \rightarrow \mathbb{T}^n$ such that $\exp \circ d\theta = \theta \circ E$. Since $B_F^*(C)$ is an open subset of $L(G)$ and $\exp : L(\mathbb{T}^n) \rightarrow \mathbb{T}^n$ is a local homeomorphism, it is easy to see that $E(B_F^*(C))$ contains an open neighborhood of e in G . Since F generates \widehat{G} , there exists an $\epsilon > 0$ such that $B_F(\epsilon) \subset E(B_F^*(C))$. This proves the lemma. \square

Proof of Theorem 4.6 Suppose A is a ρ -basis of \widehat{G} . By Proposition 4.8 A is k -regular for some $k > 0$. Clearly without loss of generality we may assume that $C < 1/k$. Let $A_1 \subset A_2 \subset \cdots \subset A$ be an increasing sequence of finite sets such that

$$\bigcup_i A_i = A, \quad A_n \subset \mathbf{Q}_k(A_{n-1}) \quad \forall n \geq 2.$$

Applying Lemma 4.9 we choose a positive $\epsilon < C$ such that for all $h \in B_{A_1}(\epsilon)$ there exists a $q \in B_{A_1}^*(C)$ satisfying $\chi(E(q)) = \chi(h)$ for all χ in A_1 . Let g be an element of $B_A(\epsilon)$. We choose a p in $B_{A_1}^*(C)$ such that $\chi(E(p)) = \chi(g)$ for all χ in A_1 . We claim that

$$|p(\chi)| < C, \quad \chi(E(p)) = \chi(g) \quad \forall \chi \in A.$$

By our choice of p this is true for all χ in A_1 . Suppose this holds for all χ in A_n . Let a_0 be an element of A_{n+1} . We choose integers n_0, n_1, \dots, n_r and $a_1, \dots, a_r \in A_n$ such that $n_0 \neq 0$, $|n_0| + |n_1| + \cdots + |n_r| \leq k$ and

$$n_0 a_0 = \sum_{j=1}^r n_j a_j.$$

Since $g \in B_A(\epsilon)$, there exists $\alpha \in \mathbb{R}$ such that $|\alpha| < \epsilon < C$ and $e^{2\pi i \alpha} = a_0(g)$. Now from the above equation it is easy to see that

$$e^{2\pi i n_0 \alpha} = a_0(g)^{n_0} = \prod_{i=1}^r a_i(g)^{n_i}.$$

Since $a_1, \dots, a_r \in A_n$, it follows that

$$e^{2\pi i n_0 p(a_0)} = \prod_{i=1}^r e^{2\pi i n_j p(a_j)} = \prod_{i=1}^r a_i \circ E(p)^{n_i} = \prod_{i=1}^r a_i(g)^{n_i}.$$

Therefore $e^{2\pi i n_0 (\alpha - p(a_0))} = 0$ i.e. $n_0(\alpha - p(a_0)) \in \mathbb{Z}$. On the other hand

$$|n_0(\alpha - p(a_0))| \leq n_0|\alpha| + |p(n_0 a_0)| \leq n_0 C + \sum_{i=1}^r n_i |p(a_i)|.$$

Now applying our induction hypothesis we see that

$$|n_0(\alpha - p(a_0))| \leq C \sum_{j=1}^r |n_j| < Ck < 1.$$

Hence $\alpha - p(a_0) = 0$, which implies that $|p(a_0)| = |\alpha| < C$ and $a_0(E(p)) = e^{2\pi i \alpha} = a_0(g)$. This proves the claim. Since A generates \widehat{G} as an abelian group, from the above claim we deduce that for any $g \in B_A(\epsilon)$ there exists a $p \in B_A^*(C)$ such that $E(p) = g$. This completes the proof. \square

4.4 Proof of Theorem B

Now we prove Theorem B for an endomorphism action ρ of an arbitrary semigroup Γ on a solenoid G . If (G, ρ) is expansive then from Proposition 4.3 and Proposition 4.4 it follows that a) and b) are satisfied. Conversely, suppose the conditions a) and b) are satisfied. By Lemma 3.1 there exists a finitely generated subsemigroup Γ_0 of Γ such that for all p in $L(G)$, the Γ_0 -orbit of p under the action ρ_e is unbounded. We construct an endomorphism action (H, σ) of Γ_0 as follows :

Since \widehat{G} has finite rank, there exists a finite set $F_1 \subset \widehat{G}$ such that $\mathbf{Q}(F_1) = \widehat{G}$. Also, from a) it follows that there exists a finite set $F_2 \subset \widehat{G}$ such that the

set $\bigcup_{\gamma} \widehat{\rho}(\gamma)(F_2)$ generates \widehat{G} as an abelian group. We define $F = F_1 \cup F_2$. Let $\widehat{H} \subset \widehat{G}$ be the subgroup generated by the set

$$A_0 = \bigcup_{\gamma \in \Gamma_0} \widehat{\rho}(\gamma)(F)$$

and let $\widehat{\sigma}$ be the Γ_0 -action on \widehat{H} defined by $\widehat{\sigma}(\gamma) = \widehat{\rho}(\gamma)|_{\widehat{H}}$. Let H denote the dual of \widehat{H} and let σ denote the endomorphism action of Γ_0 on H which is the dual of $\widehat{\sigma}$.

Claim : (H, σ) is expansive.

Since $\widehat{G} = \mathbf{Q}(F) \subset \mathbf{Q}(\widehat{H})$, it follows that the restriction map $p \mapsto p|_{\widehat{H}}$ is a Γ_0 -equivariant linear isomorphism from $L(G)$ onto $L(H)$. This implies that for every non-zero $q \in L(H)$, the σ_e -orbit of q is unbounded in $L(H)$. Also it is easy to see that A_0 is a σ -basis of \widehat{H} . Now applying Theorem 4.6 and Proposition 4.5 we see that (H, σ) is expansive, which proves the claim.

Let V be an expansive neighborhood of e in H . Let $i : \widehat{H} \rightarrow \widehat{G}$ be the inclusion map and let $\pi : G \rightarrow H$ be the dual of i . We claim that $\pi^{-1}(V)$ is an expansive neighborhood for the action ρ . To see this we choose any g in G such that the ρ -orbit of g is contained in $\pi^{-1}(V)$. It is easy to see that for any $\gamma \in \Gamma_0$,

$$\sigma(\gamma) \circ \pi(g) = \pi \circ \rho(\gamma)(g).$$

Therefore the σ -orbit of $\pi(g)$ is contained in V . Since V is an expansive neighborhood for σ , this implies that $\pi(g) = e$. In particular $\phi(g) = 1$ for all ϕ in F . For any γ in Γ , replacing g by $\rho(\gamma)(g)$ and applying the same argument we see that

$$\widehat{\rho}(\gamma)(\phi)(g) = \phi \circ \rho(\gamma)(g) = 1 \quad \forall \gamma \in \Gamma, \phi \in F.$$

Since $A = \bigcup_{\gamma \in \Gamma} \widehat{\rho}(\gamma)(F)$ generates \widehat{G} , it follows that $g = e$. This completes the proof. \square

Remark 4.10 Let Γ be the group of positive rational numbers under multiplication, \mathbb{Q} be the group of rational numbers under addition and $\widehat{\rho}$ be the action of Γ on \mathbb{Q} defined by $\widehat{\rho}(\gamma)(x) = \gamma \cdot x$. Let ρ be the dual action of Γ on $\widehat{\mathbb{Q}}$. Then from Theorem B it follows that $(\widehat{\mathbb{Q}}, \rho)$ is expansive. On the other

hand it is easy to see that for any finitely generated subgroup $\Gamma_0 \subset \Gamma$, \mathbb{Q} is not finitely generated as a $\mathbb{Z}(\Gamma_0)$ -module. Hence no finitely generated subgroup $\Gamma_0 \subset \Gamma$ acts expansively on $\widehat{\mathbb{Q}}$ under the action ρ . Since Γ is abelian, this shows that analogues of Corollary 3.2 and Corollary 3.4 do not hold when G is a solenoid.

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